The Handshake Problem

The problem: There are M men in a room and W women in another room for a morning workshop. All the men shake hands and all of the women shake hands. In the afternoon, the men and women come together and all the men shake hands with all the women.

Concerning the number of handshakes:

(A) There are always more handshakes in the morning than in the afternoon.
(B) There are always more handshakes in the afternoon than in the morning.
(C) There are no values of M and W such that the number of morning handshakes equals the number of afternoon handshakes.
(D) There are a finite number of values of M and W such that the number of morning handshakes equals the number of afternoon handshakes.
(E) There are an infinite number of values of M and W such that the number of morning handshakes equals the number of afternoon handshakes.

The Solution: Calculating the number of handshakes in a group of size N, it is a combination of N objects taken 2 at a time. $\binom{N}{2} = {}_N C_2 = \frac{N(N-1)}{2}$. In the room with M men and W women, the number of handshakes is MW as every man must shake hands with every woman.

So for instance, if there are 8 men and 6 women, the men have $\frac{8(7)}{2} = 28$ handshakes and the women have $\frac{6(5)}{2} = 15$ handshakes. So there are 28 + 15 = 43 handshakes in the morning. In the afternoon, there are 8(6) = 48 handshakes. So there are 5 more handshakes in the afternoon.

But it is possible to have more handshakes in the morning. If there are 7 men and 13 women, there are $\frac{7(6)}{2} + \frac{13(12)}{2} = 21 + 78 = 99$ handshakes in the morning and 7(13) = 91 handshakes in the afternoon. So there are 8 more handshakes in the morning.

Using an Excel spreadsheet with the number of people in the room up to 100 and more men than women, I came up with these combinations with equal handshakes in the morning and afternoon.

Women W	1	3	6	10	15	21	28	36	45	55	66	78
Men M	3	6	10	15	21	28	36	45	55	66	78	91

What I find interesting about this is the there is a pattern to the men and women. Starting with the women at 1, the numbers go up by 2, then 3 then 4, etc. The men is the same list but starting at 3. These suggest a quadratic pattern.

The pattern for women is $\frac{n(n+1)}{2}$ where n = 1, 2, 3, … or $\frac{n(n-1)}{2}$ where n = 2, 3, 4, …

The pattern for men is $\frac{(n+1)(n+2)}{2}$ when n = 1, 2, 3, … or $\frac{n(n+1)}{2}$, where n = 2 , 3, 4

So the total number of participants $M + W$ in the workshop where the number of handshakes in the morning and afternoon are the same is given by the following formula where $n = 1, 2, 3, \ldots$

$$M+W=\frac{n(n+1)}{2}+\frac{(n+1)(n+2)}{2}=\frac{(n+1)(n+n+2)}{2}=\frac{(n+1)(2n+2)}{2}=\frac{2(n+1)(n+1)}{2}=(n+1)^2$$

So based on the value of n, it seems a necessary (but not sufficient) condition for the handshakes in the morning equaling the handshakes in the afternoon is that the total number of men and women must be a perfect square.

n	1	2	3	4	5	6	7	8	9	10	11	12
Women W	1	3	6	10	15	21	28	36	45	55	66	78
Men M	3	6	10	15	21	28	36	45	55	66	78	91
$M+W=(n+1)^2$	4	9	16	25	36	49	64	81	100	121	144	169

But that is not a proof. It is true for values of M and W for values through 100. And it assumes the number of women is $\frac{n(n+1)}{2}$ and the number of men is $\frac{(n+1)(n+2)}{2}$. Gary Litvin was able to supply a proof after I first posted this, based on the statement that the total number of handshakes in the morning must equal the number of handshakes in the afternoon and I am including it. Gary's proof is better in that it does not rely on the table I created.

$$\frac{M(M-1)}{2}+\frac{W(W-1)}{2}=MW$$
$$M^2-M+W^2-W=2MW$$
$$M^2-2MW+W^2=M+W$$
$$(M-W)^2=M+W$$
$$\text{Let } n=M-W \text{ so } M+W=n^2$$

However, this doesn't prove that the pattern for women is always $\frac{n(n+1)}{2}$ and the pattern for men is always $\frac{(n+1)(n+2)}{2}$. We have shown it to be true for when $M \le 100$ and $W \le 100$ by brute force but not for when $M > 100$ or $W > 100$. Gary went a step further and showed that this pattern of numbers is the only one that yield equal number of handshakes in the morning and afternoon.

$$\text{Since } n=M-W \text{ so } M+W=n^2$$
$$\text{Adding together we get } n^2+n=2M \text{ so } M=\frac{n(n+1)}{2}$$
$$\text{Subtracting, we get } n^2-n=2W \text{ so } W=\frac{n(n-1)}{2}$$

These generate the same pattern of numbers for $n = 2, 3, 4 \ldots$ I prefer to use the formula where $n = 1, 2, 3 \ldots$ so we can talk about the nth smallest total group size.

For instance, if we wanted the 4th smallest total group size for an equal number of morning and afternoon handshakes, we could immediately say that there would have to be $(4+1)^2 = 25$ total people. $W = \frac{4(5)}{2} = 10$ and $M = \frac{5(6)}{2} = 15$. There would be $\frac{10(9)}{2} = 45$ handshakes for the women and $\frac{15(14)}{2} = 105$ handshakes for the men. That would give 150 total handshakes in the morning and obviously there would be 10(15) = 150 handshakes in the afternoon.

We can say that if n = 100, there would be $(100+1)^2 = 10{,}201$ total people. $W = \frac{100(101)}{2} = 5{,}050$ and $M = \frac{101(102)}{2} = 5{,}151$ in order to have the same number of handshakes in the morning as the afternoon. There will be both $(5050)(5151) = 26{,}012{,}550$ handshakes in both morning and afternoon.

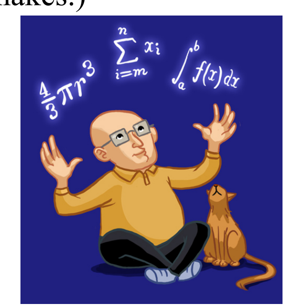
So there are an infinite number of combinations where there the number of handshakes in the morning and the afternoon are the same and the correct answer is (E).

You can also ask about the total number of handshakes in both morning and afternoon but then everyone would eventually shake hands with everyone else so the total handshakes are

$$\left(\frac{M+W}{2}\right) = {}_{M+W}C_2 = \frac{(M+W)(M+W-1)}{2}.$$

I took this a little further and investigated what would happen if there were 3 groups, men, women, and students with the possible size 25 and under. I haven't done the analysis (maybe one of the www.mastermathmentor.com members would look at it) and found that were combinations that would yield an equal number of handshakes in the morning and afternoon. (In the afternoon, with W women M men, and S students, there would be $WM + WS + MS$ handshakes.)

Women	1	1	3	3	3
Men	3	9	6	9	19
Students	9	18	19	24	15
Total Group Size	13	28	28	36	37
AM and PM handshakes	39	189	189	315	189



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