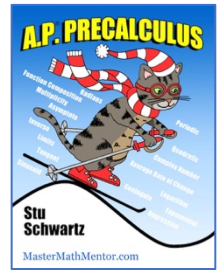
Topic 1.7 – Polynomial Modeling - Classwork



There are people who study grammar just because they enjoy it. But for most of us, the study of grammar is simply a means to an end: the ability to communicate in written form.

The same is true in mathematics. There are the rare people who enjoy math for math's sake, but for most, math is simply a way to solve problems of life that involve numbers. In grade school, we learned how to add whole numbers but quickly moved from adding 2 and 3 to adding 2 apples plus 3 apples. Most of the math that we learned with numbers quickly got translated into situations that we encounter daily. When we went to school, this process was called "solving word problems." Now we use a more sophisticated term – *modeling*.

Mathematical modeling refers to the process of creating a mathematical representation of a real-world scenario to provide insight or make a prediction. Students are usually uncomfortable with this process because it is open-ended. In chapter 1.6, we laid out a procedure for finding information about a polynomial. It is iterative – we always perform the same steps. But in modeling, every problem is different. However, there are a series of steps that we always take when we model.

- Read the problem. Do not do or write anything until you have read it and understand it.
- Identify the problem. Know what it is you want to find.
- Identify all variables – those whose value you know and those you want to find.
- Do the math. Develop a relationship between input information and output information.
- Analyze the solution. Ask yourself if the answer makes sense.
- State the answer(s). Make it clear at the end of the problem what your final answer is. Be sure to answer the question asked. Reread the problem to be sure you are doing so.

Realize that in the AP test you are taking in May, there are open-ended problems (free response). Unlike multiple-choice problems, the grader needs to know not only the answer to the problem but the procedure you used to find that answer. It must be clear and concise. The goal is the solution rather than the answer.

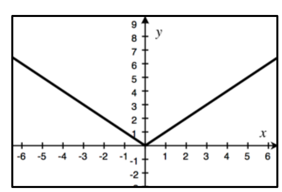
Piecewise Functions

A piecewise function is a function that has two or more rules, based on non-overlapping intervals of *x*. If *x* is this, follow this rule. If *x* is something else, follow that rule. So much of our lives are based on making a decision of what to do, based on some circumstance.

- Tax brackets: If we make this amount or less, pay percentage A in taxes. Otherwise, pay percentage B.
- Mobile phones: If you use this amount of data or less, pay this rate. Otherwise, pay that rate.
- Medication: If your blood pressure is moderately high, take this amount of medication. If it abnormally high, take this amount.

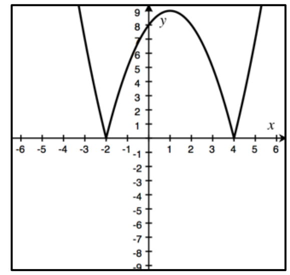
The Absolute Value Function

The most common mathematical example of a piecewise function is the absolute value function. $y = |x|$. Expressing it as a piecewise function gives $|x| = \begin{cases} x, & x \ge 0 \\ -x, & x < 0 \end{cases}$.



This gives our familiar V-shaped graph. In general, $|f(x)| = \begin{cases} f(x), & f(x) \ge 0 \\ -f(x), & f(x) < 0 \end{cases}$.

So while $y = |x|$ gives our V-shaped curve with vertex at the origin, not all functions involving an absolute value give a V-shaped curve. By definition, the absolute value function takes any graph below the x-axis and makes it positive. For instance,



So $|x^2 - 2x - 8| = \begin{cases} x^2 - 2x - 8, & x^2 - 2x - 8 \ge 0 \\ -(x^2 - 2x - 8), & x^2 - 2x - 8 < 0 \end{cases} = \begin{cases} x^2 - 2x - 8, & x \ge 4, x \le -2 \\ -(x^2 - 2x - 8), & -2 < x < 4 \end{cases}$.

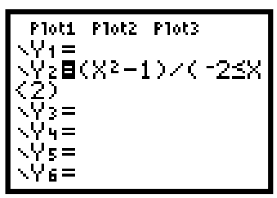
We will go into this in greater depth in our study of function transformations in section 1.11. .We also study polynomial inequalities in section 1.10.

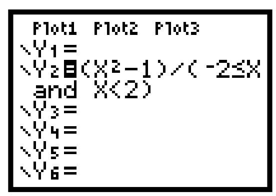
Graphing Piecewise Functions

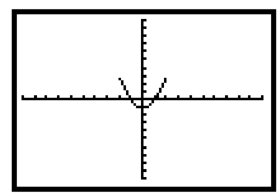
To graph a piecewise function like this on the TI-84,: $y = \begin{cases} 2x + 7, x < -2 \\ x^2 - 1, -2 \le x < 2 \\ \sqrt{5x - 1}, x \ge 2 \end{cases}$, you need three functions.

To put in Y1, we place the function in parentheses and then divide it by (X<-2). The reason we use a division sign is that the calculator evaluates values of x and generates a one (1) if X < –2 and a zero (0) if X ≥ –2. For instance if $x = -4$, the calculator is evaluating (2(–4) + 7) / (–4 < –2) or –1/1 = –1 and the point (–4, –1) is plotted. However, if $x = 1$, the calculator evaluates (2(1) + 7) / (1 < –2) or 9/0. This generates an error and nothing is plotted which is the effect we want.

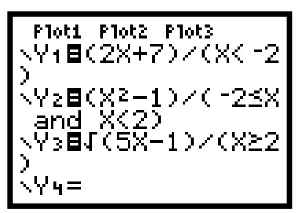
To put in Y2 with its compound statement, you may *not* do it using the technique on the left screen: While this does not generate an error, it is not what you want. Use the middle screen. You can find the operator "and" in the **2nd** **MATH** **LOGIC** menu. The screen on the right shows the graph of Y2.

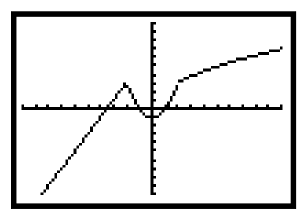






So, the entire function looks like this and the graph to the right. The problem of continuity needs to be done using calculus limit techniques. Do not trust your eyes using the calculator. When graphs are nearly vertical, the calculator has trouble graphing it. ZOOMING IN may or may not confirm the continuity of this function at $x = 2$. Finally, when TRACING, realize that there are three separate functions here and if you are tracing at an x-value in which the curve is not defined, no y-value will show.





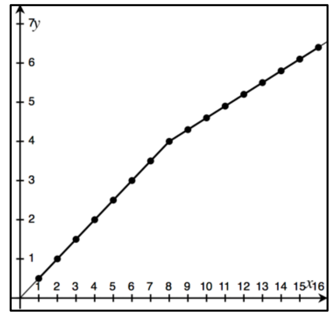
Modeling Piecewise Functions

Suppose a Philadelphia vendor is selling soft pretzels at 50 cents per pretzel. However, he has a deal that anyone buying more than 8 pretzels only pays 30 cents for every pretzel over 8.

Let's create a table that has various number of pretzels and their cost:

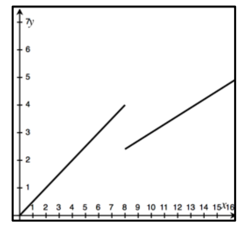
Pretzels	1	2	4	8	9	12
Cost	**0.50**	**1.00**	**2.00**	**4.00**	**4.30**	**5.20**

In creating the piecewise function that describes this situation, we define x as the number of pretzels purchased. Realize that the domain of the function is positive integers. You cannot purchase fractions of pretzels.



The first function is simple: $f(x) = 0.50x, x \le 8$. However, the second function is a bit tricky. To get the better pretzel rate for buying more than 8 pretzels , you have to first purchase 8 of them, meaning that you have to pay \$4.00. We then pay \$0.30 per pretzels for all pretzels over 8. So this can be described as $f(x) = 4 + 0.3(x-8), x > 8$. That can be simplified to $f(x) = 0.3x + 1.6, x > 8$.

Putting it all together, our piecewise function is $f(x) = \begin{cases} 0.5x, x \le 8 \\ 0.3x + 1.6, x > 8 \end{cases}$. When we graph it, realize that we get two lines, although our true graph is a series of points. But the graph is continuous at $x = 8$. We will study continuity in AP calculus, but for now, it is important to realize that in piecewise curves, we usually want the function to be drawn without taking our pencil from the paper.

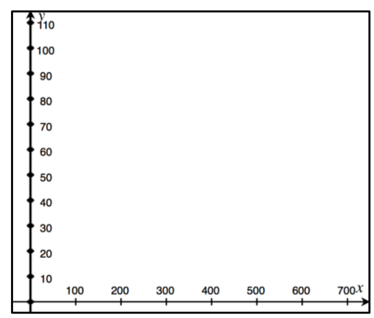


The most common error is writing the piecewise function as $f(x) = \begin{cases} 0.5x, x \le 8 \\ 0.3x, x > 8 \end{cases}$, graphed to the right. This non-continuous graph would have someone buying 10 pretzels paying less than someone who was purchasing 8.

Examples) For each situation, complete the table and write and graph a piecewise function that describes it.

1) In a small town, a home is charged a flat rate of \$10 plus 13 cents per kilowatt-hour of usage for up to 400 kilowatt hours and 11 cents for kilowatt hours of usage over 400. Complete the table.

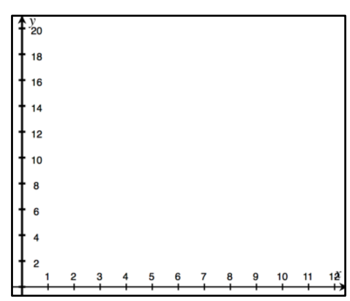
Kilowatt hrs. - x	100	200	300	400	500	600
Cost - *y*						



2) A new 12-week diet claims that it will take weight off a person according to the following formula.

In the first 2 weeks, it will take off 7 pounds at a constant rate.
In the next 4 weeks, it will take off 1.5 pounds a week at a constant rate.
In the next 6 weeks, it will take off a pound a week at a constant rate.

Week - x	1	2	4	6	8	10	12
Pounds off - *y*							



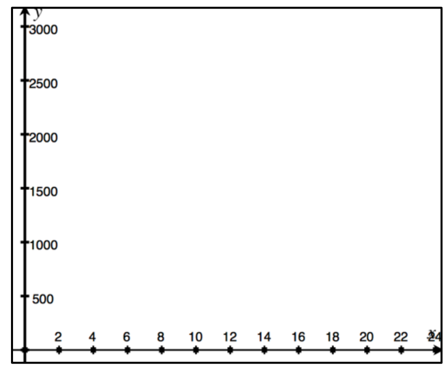
3) An estate lawyer charges according to the following:

- First hour or part cost \$50
- The next 9 hours cost \$150/hour
- After that, the charge is \$125 an hour

His partner charges differently:

- \$500 for the first 10 hours
- \$200 an hour for anything over that

Hours (x)	1	3	5	10	12	18
Lawyer 1						
Lawyer 2						



When is each lawyer cheaper to hire? Show the analysis that leads to your conclusion.

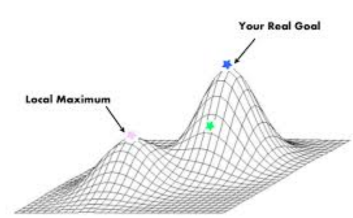
Modeling with Polynomials

In most of the problems we have tackled, our answers were "nice." We had zeros of polynomials at integer values. We had curves increasing between 2 integers. But the problems were contrived to allow easy computation. But in the real world, things don't work out that well. When you purchase something, rare is the amount you pay a whole number.

That opens up the world of the calculator. We have seen that the calculator can find zeros to as many decimals as we want. It can find relative maxima or minima, again, to extreme accuracy. So as we tackle modeling, we will be using our calculators to graph functions. We also do this because in order to solve these problems analytically, we need calculus. And that isn't until next year.

Optimization

The definition of *optimizing* is to find the "best" solution to a problem. The word "best" though is open to interpretation. The best route to travel to school may be the shortest time to get there. It could also mean the shortest distance. Those two "best routes" are not necessarily the same routes. To some people, the best cell phone to buy is the cheapest one you can find. To others, the best phone might be most quality phone on the market. Those two phones are rarely the same.



Students face these problems daily. They wish to maximize their grade while minimizing the work they do. Typically, they cannot have both at the same time.

In optimization problems, you will solve problems like:

- finding the price to charge for an object to make the most money
- determine how much should be spent on advertising to maximize sales
- building a structure using the least amount of material
- building a yard enclosing the most amount of space
- finding the least amount of medication to take to help a medical problem
- determining a speed limit that will maximize the cars per hour on a congested highway
- finding the largest area shape that can fit inside another shape

In this section, we will tackle such problems using polynomials while in topic 1.12, we will do the same with rational functions. While most of the polynomials are quadratic, some are cubic and greater.

General Method for Solving Optimization Problems with Polynomials

1. If you are given an equation that models a real-life situation, go directly to step 7.

2. Assign variables to all given quantities and quantities to be determined. Don't be afraid to use letters you usually do not use (p, m, g, etc.). When feasible make a sketch of the problem.

3. A step that helps students is make a chart of possible answers, allowing you to see a relationship between variables. While not necessary, it is helpful.

4. Write a "primary" equation for the quantity that needs to be maximized or minimized. Examples:

   - Area of rectangle = length $\cdot$ width
   - Volume of rectangular solid = length $\cdot$ width $\cdot$ height
   - Profit = income $-$ cost
   - Income = items sold $\cdot$ price per item
   - Perimeter of rectangle = $2(\text{length} + \text{width})$
   - Volume of cylinder = $\pi \cdot \text{radius}^2 \cdot$ height

5. Reduce the right side of this "primary equation" to one having a single variable. If there is more than one variable on the right, you must write a "secondary equation" (a restriction or constraint) that sets up a relationship between the variables on the right side of the primary equation.

6. In the secondary equation, solve for the easiest variable to replace into the primary equation. Do the replacement and simplify if possible.

7. Graph this function of x. Even though your equations might use different variables, graphing must be y as a function of x. This step can be tricky as there is no one to tell you the optimal viewing window. Use x-values that make sense in the context of the problem and then use a ZOOM FIT. If your function has a vertical asymptote, there is no highest or lowest value of y and you will have to adjust your Ymin and YMax.

   Use your calculator's built-in keys to find the absolute maximum or minimum value, zeros, or intersections.

8. Be sure to answer the question that is asked. For instance, there is a difference between asking for the price to charge for an item to maximize your profit as opposed to asking for the maximum profit. Be sure to include units of measure in your answers.

Some of the examples below are the same as you will see next year when you will solve them analytically using calculus rather than graphically. Sketch the graph that represents the problem situation.

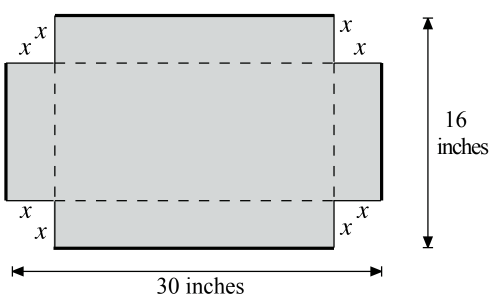
1) Two numbers sum to 40. Find the numbers that minimize and maximize their product. Complete the table.

Smaller number	1	2	4	0.5	–10	x
Larger number						y
Product						xy

2) A rectangle has a perimeter of 71 feet. What is the minimum and maximum area of the rectangle?

Width	10	5	1	0.5	0.1	x
Length						y
Area						

3) An open box is to be made from a piece of metal 16 by 30 inches by cutting out squares of equal sides from the corners and bending up the sides. What size square should be cut out to create a box of greatest volume and what is the maximum volume? Is there a minimize volume? Complete the table.



x	1	2	3	4	6	x
Length						
Width						
Volume						

4) A trucking company has determined that the cost per hour to operate a single truck is given by $C(s) = 0.1s^2 - 10s + 450$ where s is the speed that the truck travels. At what speed is the cost per hour a minimum? What is the hourly cost to operate the truck at that speed? Assuming that the truck can travel from 0 to 75 mph, what is the range of hourly cost to operate the truck?

5) Normally a pear tree will produce 30 bushels of pears per tree when 20 (or fewer) pear trees are planted per acre. However, for each additional pear tree above 20 trees per acre, the yield per tree will fall by one bushel per year. How many trees should be planted per acre to maximize the total yield?

additional tree	Trees	Bushels	Pears
0			
1			
2			
30			
x			

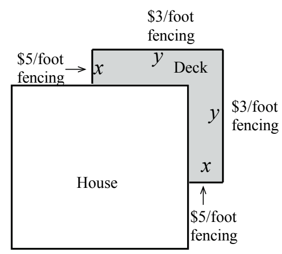
6) A real estate company owns 100 apartments in Chicago. At $2,100 per month, each apartment can be rented. However, for each $75 increase in rent, there will be two additional vacancies. How much should the real estate company charge for rent to maximize its revenues? Complete the table.

$75 increases	Rent	Apts rented	Revenue
0			
1			
2			
3			
4			
50			
x			

7) A company makes high end specialty vehicles. Its facilities allow for a maximum of 50 cars annually to be built. The profit P for selling x cars is given by $P = -73x^3 + 4425x^2 - 225000$. Using this model, approximate the number of vehicles they should build in order to have a profit of $2 milion.

8) Mr. Tyree is building a symmetric deck of constant width on the corner of his house as shown in the figure to the right. Fencing is placed on the edges of the deck with no fence needed against the house. Fencing perpendicular to the house costs $5/foot while fencing parallel to the house costs $3/foot.

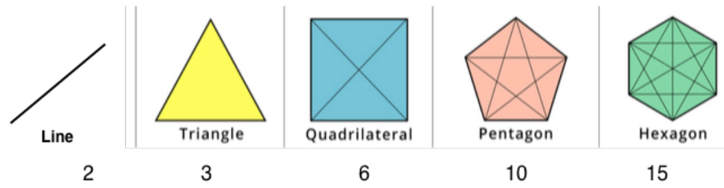
If Mr. Tyree wants to spend no more than $250, find the maximum area of the deck.



9) A company sells electronic bikes. The total revenue R in thousands of dollars is related to the number of bikes them make annually x by the function $R = \frac{1}{75000}\left(500x^2 - x^3\right), x \le 500$. a) Determine how many bikes they should make to maximize revenue and what that revenue is and b) approximate the number of bikes they should make when the revenue is increasing the fastest.

Problem 9 has this neat formula to predict revenue from bikes sold but things rarely work according to formulas. Typically real-life problems use data rather than equations. We can use the successive difference method to predict future data.

Suppose a country has n major cities and airlines connect every city with every city. We are interested in the number of airline routes there are. In order to picture this, let's use regular polygons with cities at every vertex. We are interested in how many connecting lines there are. We count them and then create successive differences.



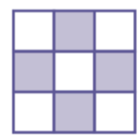
We assume that since the entries on the 2nd successive difference is always 1, that this data was formed by a quadratic. So now, in order to find how many routes are available with 7 cities and 8 cities, we work backwards. As seen below, 7 cities has 21 routes and 8 cities has 28 routes.

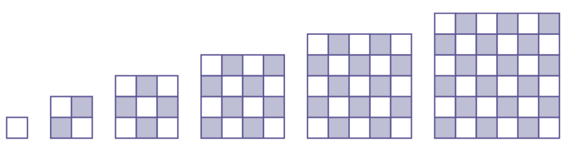
Cities	2		3		4		5		6
Routes	1		3		6		10		15
		2		3		4		5	
			1		1		1		

Cities	2		3		4		5		6		**7**		**8**
Routes	1		3		6		10		15		**21**		**28**
		2		3		4		5		**6**		**7**	
			1		1		1		**1**		**1**		

Of greater benefit is developing the quadratic equation that defines the number of routes. Knowing that this is a quadratic, use analytic methods taught in algebra to find the equation passing through any 3 points. Or, quadratic regression on your calculator can be used. (The number of routes is: $R = 0.5n^2 - 0.5n$.)

10) We define S to be the total number of squares on an n by n checkerboard. For instance, on a 3 by 3 board, there are 9 squares that are 1 by 1, 4 squares that are 2 by 2 and 1 square that is 3 by 3. So $S = 14$. Use the picture below to determine the total number of squares on a chessboard (8 by 8).





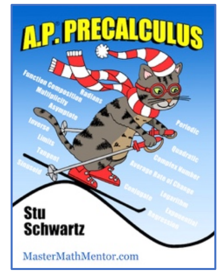
n	1	2	3	4	5	6
S	1	5	14	30	55	91

11) A town in Japan discovered that it had 2 Komodo dragons in the year 2016. Komodo dragons are asexual meaning that the female can reproduce all be herself. The town officials began to track the number of Komodo dragons (D) each year after 2016 (n). Predict the number of Komodo dragons that will be in the town in the years 2024 and 2025.

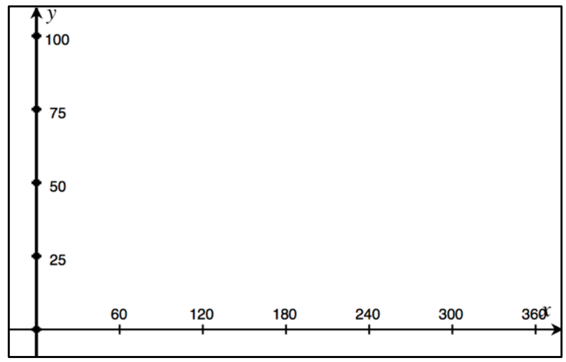
n	0	1	2	3	4	5	6	7
D	3	7	13	23	38	59	87	122

Note that the data isn't perfect. The 3rd set of difference are not all equal to one. Rarely does data exactly fit polynomials. This is just a prediction that in 2024, the town will have 217 dragons and in 2025, 269 dragons if the cubic trend continues.

Topic 1.7 – Polynomial Modeling - Homework



1. A casino gives players an electronic card that keeps track of how long they are gambling. Based on the amount of time gambling, players get "comped": given rewards for playing longer (and presumably losing). If a player spends between one and two hours gambling, he gets comped a free meal worth $25. After 2 hours, he gets the meal and is also given a comp worth 30 cents for every minute above 2 hours. After 5 hours, he is given a comp worth 50 cents for every minute after 5 hours. Write and graph (up to 6 hours of play) a piecewise function that describes the comp y as a function of time t. What is the average rate of change of the comp between 2 and 6 hours of play?



2. When purchasing a cell phone, a customer has a choice of 2 phone plans.

Plan 1:

- Pay $25 a month
- Pay 10 cents a minute up to 30 minutes
- Pay 8 cents a minute for all minutes over 30

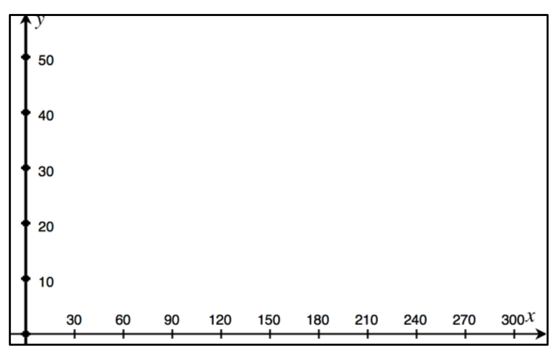
Plan 2:

Pay 20 cents a minute for up to 120 minutes
Pay 15 cents a minute for all minutes over 120

Complete the table:

Minutes	**0**	**10**	**20**	**30**	**60**	**100**	**120**	**180**	**300**
Plan 1									
Plan 2									

Write piecewise functions for plan 1: $f(t)$ and plan 2: $g(t)$ and graph both.

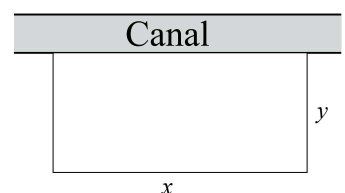


What is your recommendation for the best (cheapest) plan and why?

3. Find two numbers whose sum is 10 for which the sum of the squares is a minimum. How about a maximum?

4. Find nonnegative numbers x and y whose sum is 75 and for which the product xy^2 is as large as possible. Is it possible for the product xy^2 to be as small as possible? Explain.

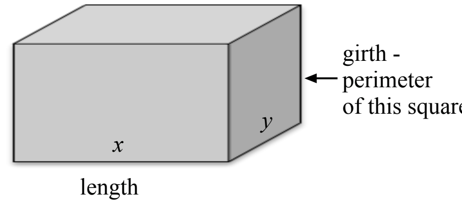
5. A farmer has 2,000 feet of fencing to enclose a pasture area. The field will be in the shape of a rectangle and placed against a canal where no fencing is needed. What is the largest area pasture than can be created and what are its dimensions?



6. A fisheries biologist is stocking fish in a lake. She knows that when there are n fish per unit of water, the average weight of each fish will be $W(n) = 500 - 2n$, measured in ounces. What is the value of n that will maximize the total fish weight per unit of water? Complete the table and find the total fish weight.

n	0	1	10	50	200
$W(n)$					
Weight of fish					

7. The U.S. Postal Service will accept a box for domestic shipping only if the sum of the length and girth (distance around the box) does not exceed 108 inches. Find the dimensions of the largest volume box with a square end that the USPS will accept. What is the largest volume box allowable measured in cubic feet? For approximately what volume box is the price increasing the fastest?



8. Blood pressure in a patient will drop by an amount D where $D = 0.025x^2y$, where x is the amount of medication injected in cm^3 and y is the amount of air in the 30 cm^3 syringe. Find the dosage that provides the greatest drop in blood pressure. What is the drop in blood pressure? At approximately what dosage is the decrease the fastest?

9. The profit for an online company is given by $P = 25000 - a^3 + 30a^2$ where a is the amount spent (in hundreds of dollars) on advertising. What amount of advertising yields the maximum profit and what is that profit? Explain why if they spend more than a, their profit may not be as great. The point of *diminishing returns* is the amount spent in advertising where the profit is rising the fastest. Approximate this point.

10. A national muffler shop charges $120 to replace a muffler. At this rate, the company replaces 7,500 mufflers per week. For each additional 5 dollars the company charges, it tends to lose 200 customers a week. How much should the company charge in order to maximize their revenue? (If confused, make a table like #6 classwork problem).

11. Market research tells you that if you set the price of a pack of gum at $1.50, you will be able to sell 5,000 packs. For every 10 cents you lower the price below $1.50, you will sell 1,000 more packs. Suppose your fixed costs is $2,500 and each pack of gum cost you $0.20 to make. What should be the price of a pack of gum to maximize profit?

12. An island resort has 75 rental units. When the rent is $1,300 a week, all units are occupied. However, on the average, for each $50 increase in weekly rent, 2 units become vacant. Each occupied unit requires an average of $72 a week for service and repairs and each un-occupied unit requires $350 a week to keep minimum air-conditioning in the unit. What rent should be charged to realize the most profit?

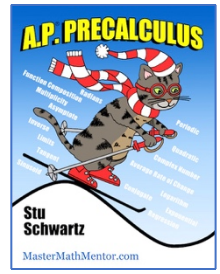
13. An oriental rug maker creates circular rugs out of silk. He charges based on the diameter of the circle based on the table to the right. What do you predict an 8-foot diameter rug would cost?

Diameter (ft)	1	2	3	4	5	6
Cost	$28	$224	$756	$1,792	$3,500	$6,048

14. The population of Manhattan Island has changed radically over the past years. At the right is a table showing the population from 1900 to 2000. Let *n* be the number of 20-year periods since 1900. Let *P* be the approximate population in millions of people at these times. Use differences to predict what the population was in 2020 and what it will be in 2040.

Year	1900	1920	1940	1960	1980	2000
Population	1,880	2,185	2,015	1,660	1,410	1,555

Topic 1.8 – Rational Functions – Classwork



We have spent a good deal of time working with polynomial functions. None of these functions had denominators, other than possible constants. We now turn to rational functions in the form of $\frac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are polynomial functions, $q(x) \neq 0$. Unlike polynomials, these graph very different curves. The behavior of the rational function is affected the most by the polynomial with the greater degree.

For purposes of this document, let's use the following definitions:

Bottom-heavy –	the degree of $q(x)$ is higher than the degree of $p(x)$
Top-heavy –	the degree of $p(x)$ is higher than the degree of $q(x)$
Powers-equal –	the degree of $p(x)$ is the same as the degree of $q(x)$

Before we go through the mechanics of working with these type of functions, let's do some analysis to understand why their graphs have the behavior they do. Let us look at $f(x) = \frac{x+1}{x}$. This is a "powers-equal" function as the the highest power of x (whch is 1) appears in both numerator and denominator.

We can "split" the fraction and write it as $f(x) = \frac{x}{x} + \frac{1}{x} = 1 + \frac{1}{x}$. Think of it as writing an improper fraction as a mixed number. So, our function is equal to one plus a little bit more. It should be clear as shown by the table below that when x is large or small, the effect that $\frac{1}{x}$ has on the value of the function is negligible. And as x gets much larger or smaller, the effect gets smaller yet, as to be insignificant.

x	10	100	1000	10000	–10	–100	–1000	–10000
$f(x)$	1.1	1.01	1.001	1.0001	0.9	0.99	0.999	0.9999

We can express this concept using limits. In our study of polynomials, we used limits for end behavior. This graph however has a different end behavior – the further to the right or left we go, the closer the function gets to the value of 1. This is expressed as $\lim\limits_{x\to\infty} f(x) = 1$ and $\lim\limits_{x\to-\infty} f(x) = 1$. Note that the value of the function will never be 1. We are always adding or subracting an amount, even though it might be miniscule.

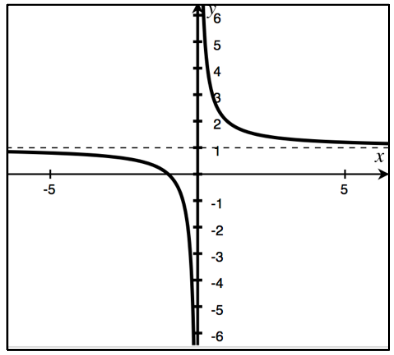
Realize that x cannot equal zero. So the question is, what happens to the value of $f(x)$ when x gets close to zero? Again, we use a table to explore.

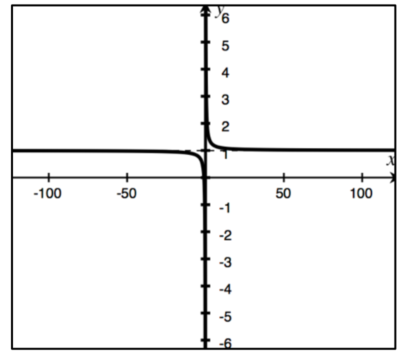
x	0.1	0.01	0.001	0.0001	–0.1	–0.01	–0.001	–0.0001
$f(x)$	11	101	1001	10001	–9	–99	–999	–9999

Two things should be apparent. First, the signs of $f(x)$ are different depending from which direction we approach $x = 0$. If we approach from the right side, the values are positive. If we approach from the left side, the values are negative. Secondly, the values are getting infinitely large or infinitely small. We express this using limits: $\lim\limits_{x\to 0} f(x) = \infty$ and $\lim\limits_{x\to 0^-} f(x) = -\infty$.

So, the best description of the graph of $f(x)$ is "the graph is essentially 1 except when x is close to zero when it explodes upwards or downwards."

It certainly doesn't look like the line $y = 1$. Let's look at it on a different window to convince you.

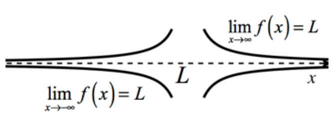




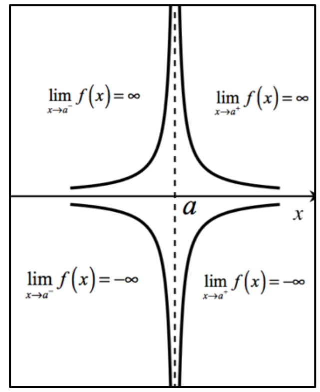
Note that we have a dashed line at $y = 1$. This is called a *horizontal asymptote*. While not a part of the actual graph, it is placed on the graph to show that the value of $f(x)$ gets infinitely closer to that line, the larger or smaller the value of x. This is another way of expressing end behavior.

This curve also has a *vertical asymptote*. It is the y-axis (the line $x = 0$). We usually show vertical asymptotes with dashed lines as well but since in this case the asymptote is the axis itself, we de no not.

In this section, we will concentrate on bottom-heavy and powers-equal expressions. Top-heavy expressions will be tackled in topic 1.8. Different than topic 1.6, where we attempted to find the zeros, we are interested in actually graphing rational functions by finding important information. There are a series of definitions and steps to follow which will give us the necessary information to graph the function on the supplied graph.

Term	**Definition**	**How to find it**
a. Domain Range	the set of values that x can take. Usually we express the domain in terms of values x cannot take. The set of values y can take. It is not usually asked as it has little bearing on the graph.	Set the denominator equal to zero. The roots of the denominator are what x *cannot* be. If the denominator is never equal to zero, then the domain is all real numbers. After you do this, factor both numerator and denominator and do any cancellation. From now on, it is best to work with this rational function.
b. Vertical Asymptotes (VA)	A vertical line in the form of $x = k$ that the curve approaches but never reaches. VA's are usually shown by a dashed vertical line, except on the y-axis on your graph. (see below)	Set the denominator equal to zero as VA's are x-values where the denominator equals zero. There can be more than one VA. If the denominator can never equal 0, then the function has no VA.
c. Horizontal Asymptote (HA). This is a term to describe end behavior. $\lim\limits_{x\to\infty} f(x)$ and $\lim\limits_{x\to-\infty} f(x)$ 	A horizontal line in the form of $y = L$ that the curve approaches the further right or left you go on the graph. Unlike the VA, it is possible for a graph to cross its HA. Usually shown by a dashed horizontal line, except on the x-axis. If L is the H A, then $\lim\limits_{x\to\infty} f(x) = L$ and $\lim\limits_{x\to-\infty} f(x) = L$	Rational functions that are bottom-heavy or powers equal always have an HA. If the function is bottom-heavy, the HA is always $y = 0$. If the function is powers-equal the HA will be $y = \dfrac{\text{coefficient of numerator's highest power}}{\text{coefficient of denominator's highest power}}$
d. Zeros (Roots)	Where the function crosses the x-axis (if at all).	Set the numerator equal to zero. If it cannot equal zero, then there are no zeros and the function never crosses the x-axis.

e. Interval Work	A sign chart that determines whether the function is positive (above the axis) or negative (below the axis) at critical intervals. It is best to have both numerator and denominator in factored form.	Your critical values are any x-values you found in b (VA) and d (Zeros) above. Make a number line with those critical values marked. Then determine the *sign* of the function in the intervals created by plugging in sample numbers to the function. Having this information tells you for what values of x the function is above and below the x-axis.
f. y-intercept	Where the function crosses the y-axis, if at all.	Set x equal to zero.



Before we give some examples, let's go into more depth about vertical asymptotes. A vertical asymptote occurs when the values of the polynomial in the denominator are arbitrarily close to zero. If a is the value of vertical asymptote, then as we approach a from the left side, the graph either increases without bound or decreases without bound. Mathematically, we say that $\lim\limits_{x\to a^-} f(x)=\infty$ or $\lim\limits_{x\to a^-} f(x)=-\infty$. Also, as we approach a from the right side, the graph either increases without bound or decreases without bound. Mathematically, we say that $\lim\limits_{x\to a^+} f(x)=\infty$ or $\lim\limits_{x\to a^+} f(x)=-\infty$. So as we approach the vertical asymptote from either the left side or the ride side, our answer is some type of infinity, and not necessarily the same. The picture to the right shows the 4 possibilities. A function can never cross its vertical asympote as it would fail the vertical line test.

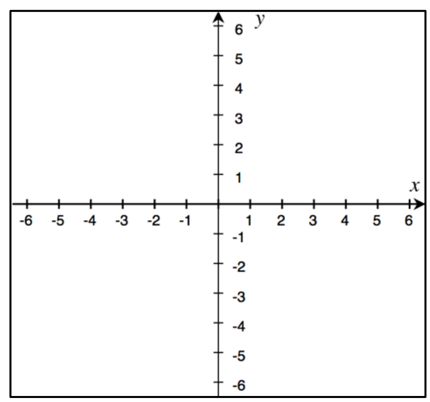
To graph the function, use the sign chart to determine whether the graph approaches the VA from the high or low side. Then it's a matter of playing "connect the dots."

Example 1) Graph the following by first finding the important features. Calculator verify.

a) $y=\dfrac{x-4}{x+2}$

a. domain:
b. VA:
c. HA:
d. Zeros
e. Interval Work: ____________________

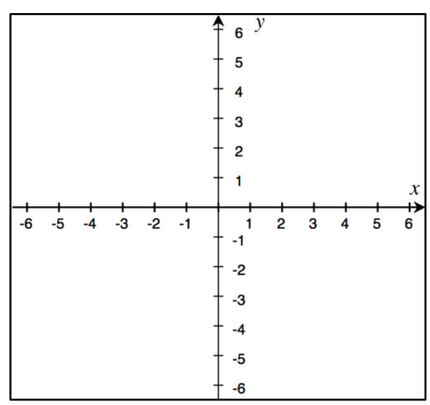
f. $y-$intercept:



b) $y=\dfrac{-4}{x-3}$

a. domain:
b. VA:
c. HA:
d. Zeros
e. Interval Work: ____________________

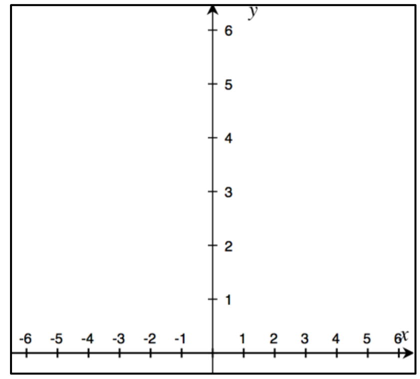
f. $y-$intercept:



c) $y = \dfrac{2-x}{x^2+4x+4}$

a. domain:
b. VA:
c. HA:
d. Zeros
e. Interval Work: ____________________

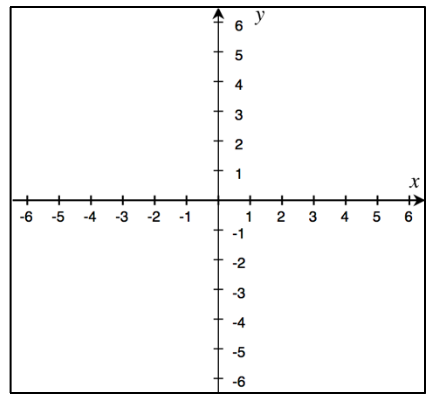
f. $y-$intercept:



d) $y = \dfrac{x}{x^2-16}$

a. domain:
b. VA:
c. HA:
d. Zeros
e. Interval Work: ____________________

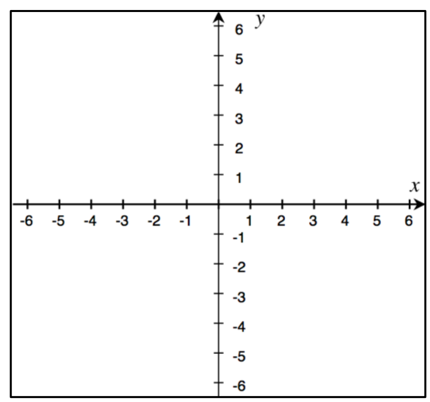
f. $y-$intercept:



e) $y = \dfrac{x^2-16}{x^2-4}$

a. domain:
b. VA:
c. HA:
d. Zeros
e. Interval Work: ____________________

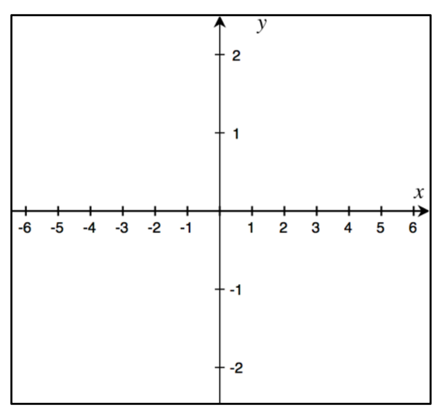
f. $y-$intercept:



f) $y = \dfrac{x^2-2}{x^2+4}$

a. domain:
b. VA:
c. HA:
d. Zeros
e. Interval Work: ____________________

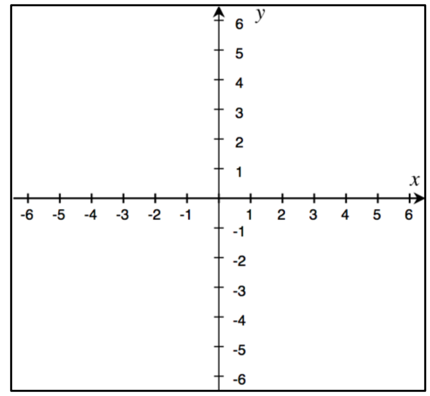
f. $y-$intercept:



g) $y = \dfrac{x^2+4}{1-x^2}$

a. domain:
b. VA:
c. HA:
d. Zeros
e. Interval Work: ____________________

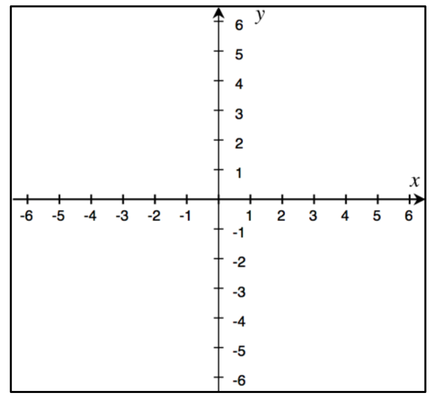
f. $y-$intercept:



h) $y = \dfrac{2x^2-2}{x^2-3x}$

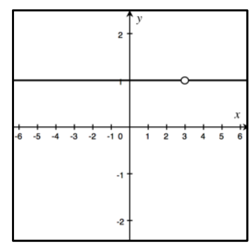
a. domain:
b. VA:
c. HA:
d. Zeros
e. Interval Work: ____________________

f. $y-$intercept:



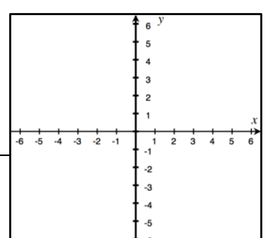
Holes

If the multiplicity of a zero z in the numerator is greater than or equal to that of the same zero z in the denominator, then the graph of the rational function has a hole at corresponding value of x. For instance, if $y = \dfrac{x-3}{x-3}$, this graphs a horizontal line at $y = 1$. However, since 3 is not in the domain, the graph has a hole at $x = 3$. The graph is to the right. So you should find the domain first before canceling and then checking out the vertical asymptote. (Note: your calculator will not show this hole when graphing).



Example 2) Graph the following by first finding the important features.

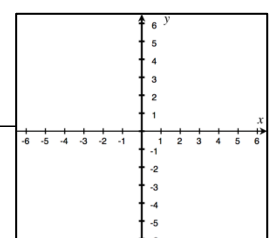
a) $y = \dfrac{x^2+3x}{x^2+5x+6} \quad = \dfrac{x}{x+2}$



a. domain:
b. VA:
c. HA:
d. Zeros
e. Interval Work: ____________________

f. $y-$intercept:

b) $y = \dfrac{-9x}{x^3-9x} \quad y = \dfrac{-9}{x^2-9}$



a. domain:
b. VA:
c. HA:
d. Zeros
e. Interval Work: ____________________

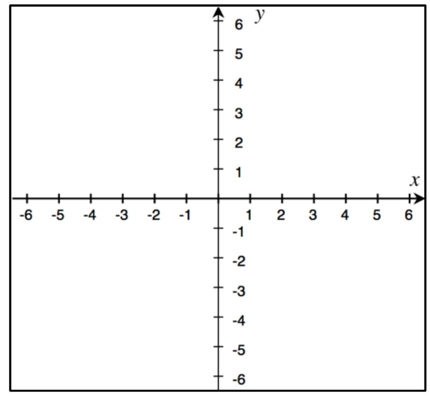
f. $y-$intercept:

Topic 1.8 – Rational Functions – Homework

1. Graph the following by first finding the important features. Calculator verify.

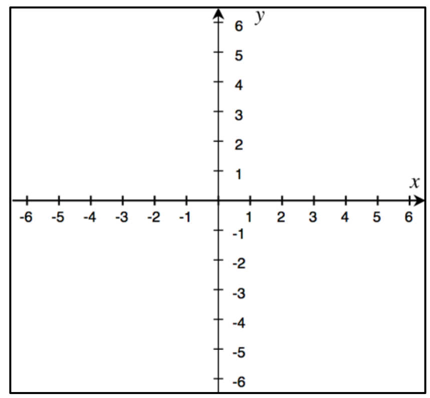
a. $y = \dfrac{3}{x+2}$

a. domain:
b. VA:
c. HA:
d. Zeros
e. Interval Work: ____________________
f. $y-$intercept:



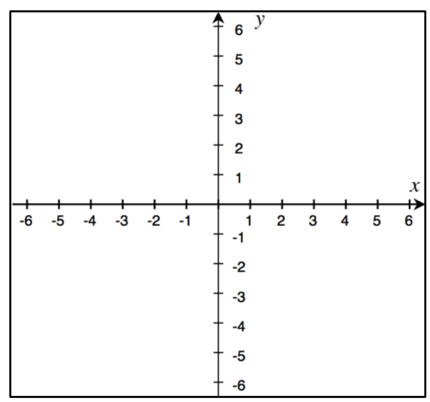
b. $y = \dfrac{3x^2-3x}{x^2+x-2} = \dfrac{3x(x-1)}{(x+2)(x-1)} = \dfrac{3x}{x+2}$

a. domain:
b. VA:
c. HA:
d. Zeros
e. Interval Work: ____________________
f. $y-$intercept:



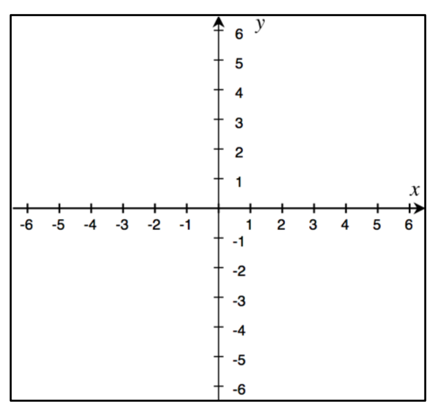
c. $y = \dfrac{x}{x^2+2x-3}$

a. domain:
b. VA:
c. HA:
d. Zeros
e. Interval Work: ____________________
f. $y-$intercept:



d. $y = \dfrac{x^2}{x^2-5}$

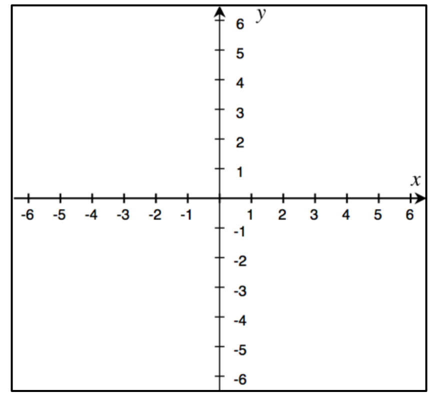
a. domain:
b. VA:
c. HA:
d. Zeros
e. Interval Work: ____________________
f. $y-$intercept:



e. $y = \dfrac{x^2 - 3x - 4}{x^2}$

a. domain:
b. VA:
c. HA:
d. Zeros
e. Interval Work: ____________________

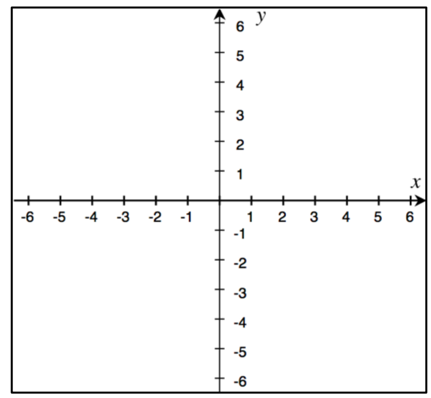
f. $y-$intercept:



f. $y = \dfrac{x^2 - 2x + 1}{x^2 + 2x - 8}$

a. domain:
b. VA:
c. HA:
d. Zeros
e. Interval Work: ____________________

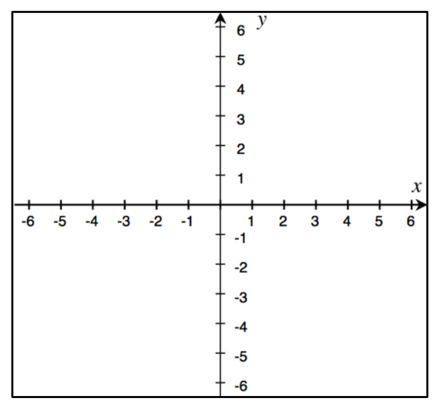
f. $y-$intercept:



g. $y = \dfrac{-4x}{x^2 - x - 12}$

a. domain:
b. VA:
c. HA:
d. Zeros
e. Interval Work: ____________________

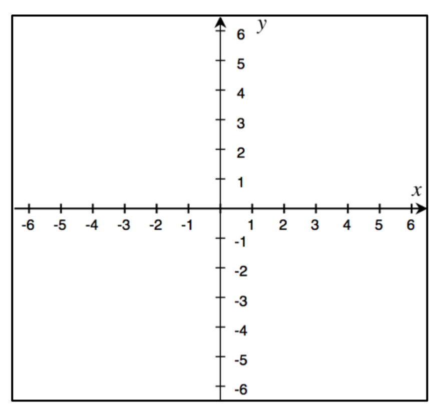
f. $y-$intercept:



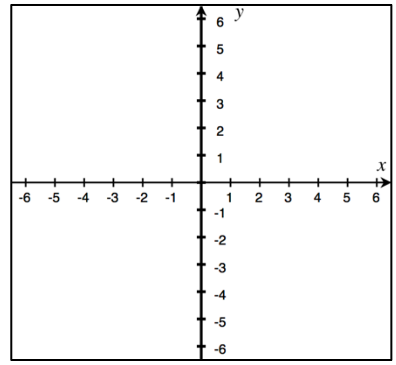
h. $y = \dfrac{4x^2 + 8x}{x^3 + 4x}$

a. domain:
b. VA:
c. HA:
d. Zeros
e. Interval Work: ____________________

f. $y-$intercept:



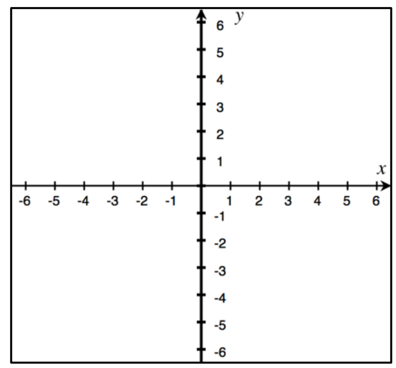
i. $y = \dfrac{x^2+4}{x^2+1}$



a. domain:
b. VA:
c. HA:
d. Zeros
e. Interval Work: ____________________

f. $y-$intercept:

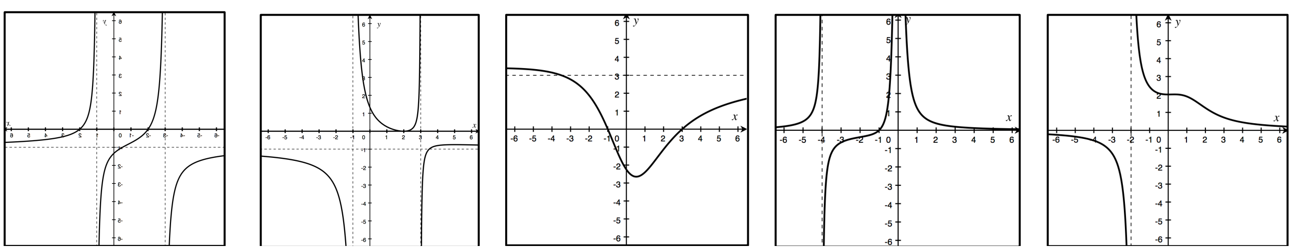
j. $y = \dfrac{1}{x} - \dfrac{x}{x+2}$



a. domain:
b. VA:
c. HA:
d. Zeros
e. Interval Work: ____________________

f. $y-$intercept:

2. Match the equations with the graphs. Calculator verify.



i. ii. iii. iv. v.

a. $y = \dfrac{3x^2-6x-9}{x^2+4}$ b. $y = \dfrac{x^2+16}{x^3+8}$ c. $y = \dfrac{4x-4-x^2}{x^2-2x-3}$ d. $y = \dfrac{3x+3}{x^3+4x^2}$ e. $y = \dfrac{4-x^2}{x^2-2x-3}$

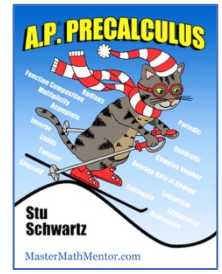
3. Write a rational function satisfying the following criteria. Calculator verify.

a. VA at $x=-2$
HA at $y=-2$
Zero at $x=2$

b. VA at $x=-2$
HA at $y=-2$
No zero

c. No VA
HA at $y=-2$
Zero at $x=2$

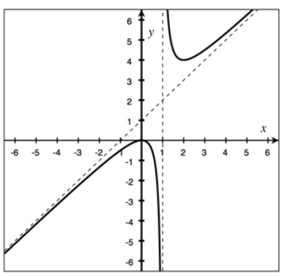
Topic 1.9 – Rewriting Expressions – Classwork

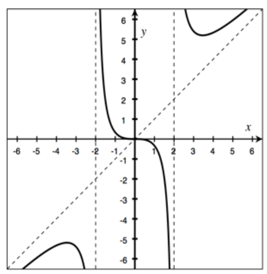


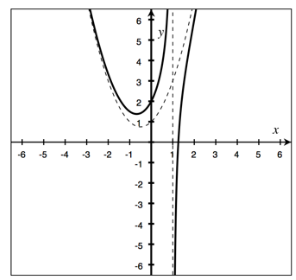
In precalculus, many times it is helpful to rewrite an expression in another form. We have seen this with lines. The general equation $3x + 2y - 5 = 0$ is more esthetically pleasing than $y = \frac{-3}{2}x + \frac{5}{2}$ but this form gives us the slope and y-intercept, making it easy to graph. In our study of polynomials, factored form allows us to easily find zeros. So we now examine alternate forms of some rational and other expressions.

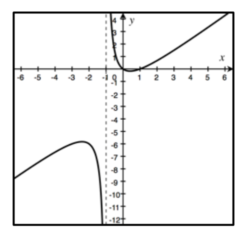
Oblique Asymptotes

We examine rational functions $\frac{p(x)}{q(x)}$ that are top-heavy. The degree of polynomial p is greater than the degree of polynomial q. When rational functions are top heavy, there is one major difference in the shape of the graphs. Instead of a horizontal asymptote, the function has an *oblique asymptote* (OA) or sometimes called a *slant asymptote*. An OA operates like a horizontal asymptote except that instead of being horizontal, the line is on a slant. Below are some graphs with OA's which are shown with dashed lines. It is also possible for an OA to be not a line, but a curve as shown in the last graph.

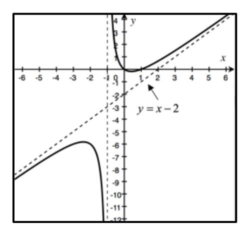

one VA and one OA


two VA's and one OA


one VA and one OA (a parabola)



For instance, let's look at $y = \frac{x^2 - x}{x+1}$ as shown in the figure to the right. This clearly has a vertical asymptote at $x = -1$ and our previous method of determining that still holds: set the denominator equal to zero. But now, it seems clear that there is no horizontal asymptote. Rather it seems that the graph goes up to the right forever and down to the left forever. That is: $\lim\limits_{x \to \infty} f(x) = \infty$ and $\lim\limits_{x \to -\infty} f(x) = -\infty$.



That is true. But there is a little more to what is occurring. The curve is becoming asymptotic to the slanted line $y = x - 2$ as shown to the right. This line is called an oblique (or slant) asymptote. Like vertical asymptotes, the function will not cross it but as x gets very large and very small and smaller, the graph of the function gets closer to this line.

$$\begin{array}{r} 3 \\ 2\overline{)7} \\ \underline{6} \\ 1 \end{array} \qquad \begin{array}{r} 29 \\ 11\overline{)326} \\ \underline{22} \\ 106 \\ \underline{99} \\ 7 \end{array}$$

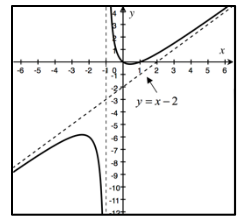
The question you should have is why is that happening and how do we find this new line? The two questions go hand in hand. In elementary arithmetic, you learned that fractions with a numerator greater than the denominator was called an improper fraction. Before you learned to work with improper fractions, you usually changed them to mixed numbers. For instance, $\frac{7}{2} = 3\frac{1}{2}$ or $\frac{326}{11} = 29\frac{7}{11}$. How did you do this calculation? You did long division, dividing 3 into 7 or 11 into 326. In each case, you had a remainder.

The same technique can be done using rational expressions. For $\dfrac{x^2 - x}{x+1}$ we can divide $x + 1$ into $x^2 - x$ and express our improper rational expression (because the degree of the numerator is greater than the degree of the denominator) as a mixed expression. And because we previously learned the technique of synthetic division, our work is easier than if we were forced to perform long division. Remember the technique of synthetic division. Since we are dividing by $x + 1$, we place a -1 to the outside, and write down our coefficients of the numerator. Do not forget the 0 term at the end. We drop the 1 and do a series of multiplications and additions. The last number is the remainder. In these cases, it will be non-zero. If it is zero, the denominator goes into the numerator evenly. That means that the expression will factor and denominator will be gone and we are graphing a polynomial, albeit one with a hole in it.

$$\begin{array}{r|rrr} -1 & 1 & -1 & 0 \\ & & -1 & 2 \\ \hline & 1 & -2 & 2 \end{array}$$

So we can now express our top-heavy rational expression in an alternate fashion. $y = \dfrac{x^2 - x}{x+1} = x - 2 + \dfrac{2}{x+1}$.

The oblique (slant) asymptote is $y = x - 2$ while the vertical asymptote is $x = -1$. Here is the explanation of the graph. The further the x value gets away from $x = 1$, the less impact the fraction $\dfrac{2}{x+1}$ has on the value of y. For instance, if $x = 99$, then $y = 97 + 0.02$, barely larger than 97. However, the closer x is to -1, the more impact the fraction $\dfrac{2}{x+1}$ has on the value of y. For instance, if $x = -1.1$, $y = -3.1 - 20 = -23.1$. I like to think of the vertical asymptote as a magnet. The closer x gets to this magnet, the more impact the asymptote has. But when x is far away from the magnet and the graph of the function is pretty much $y = x - 2$.



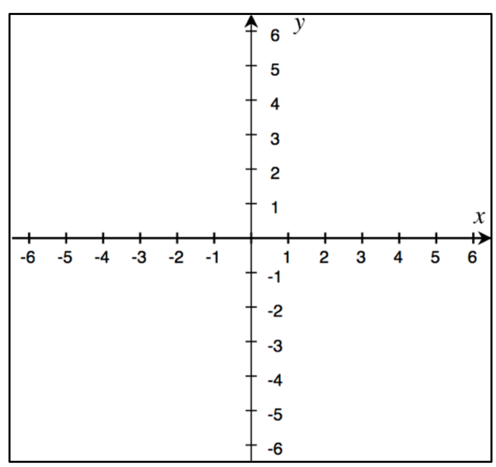
So, in general, if $h(x) = \dfrac{f(x)}{g(x)}$, then by long (synthetic) division, $\dfrac{f(x)}{g(x)} = q(x) + r(x)$, where q is the quotient, r is the remainder, and the degree of r is less than the degree of g.

Graphing top-heavy expressions is similar to the technique in graphing bottom-beavy or power-equal rational expressions. We still find the domain, vertical asymptote(s), zeros, y-intercept and do interval work. The only difference is that we do synthetic division to find the oblique asympote (OA). End behavior is that of the OA.

1) For each of the following rational functions, find the requested information and sketch. Calculator verify.

a) $y = \dfrac{x^2 - x - 6}{x - 1}$

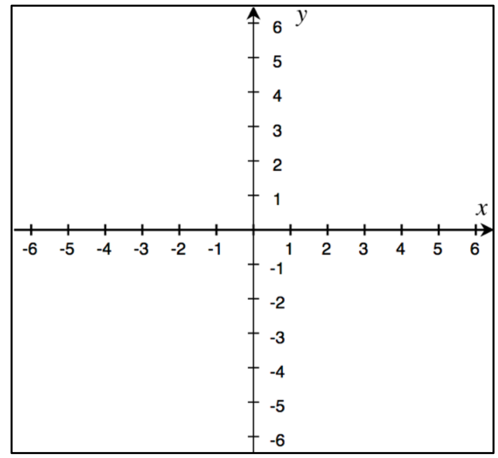
a. domain:
b. VA:
c. OA:
d. Zeros
e. Interval Work: ________________
f. $y-$intercept:



b) $y=\dfrac{x^2+2x+1}{x+2}$

a. domain:
b. VA:
c. OA:
d. Zeros
e. Interval Work: ________________

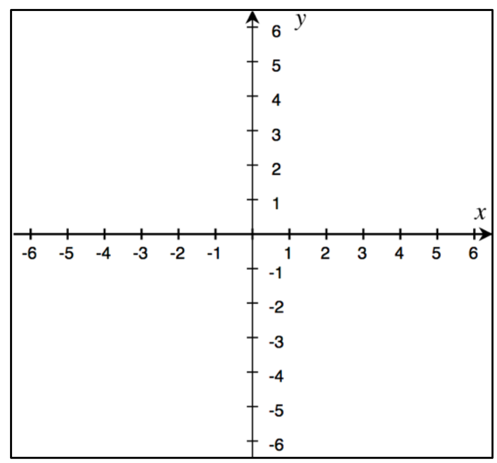
f. $y-$intercept:



c) $y=\dfrac{x^3-4x^2-11x+30}{x^2-16}$

a. domain:
b. VA:
c. OA:
d. Zeros
e. Interval Work: ________________

f. $y-$intercept:

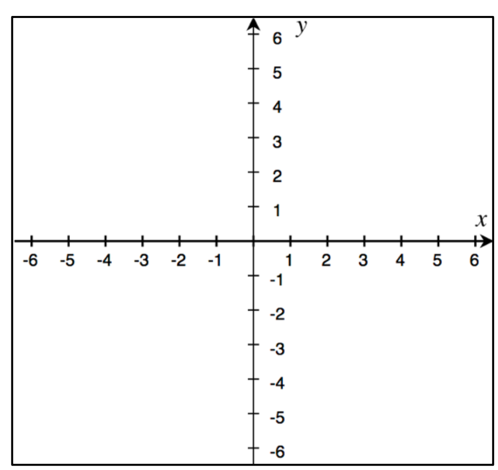


Sometimes, the numerator's degree exceeds the denominator's degree by 2. In that case, the result of the synthetic division will be a quadratic and the oblique asymptote is a parabola. Again, when the x-value is close to the VA, it will act as a magnet, but the farther away x is from the VA, the graph resembles a parabola.

d) $y=\dfrac{x^3}{x-1}$

a. domain:
b. VA:
c. OA:
d. Zeros
e. Interval Work: ________________

f. $y-$intercept:



Partial Fraction Decomposition

You are used to adding two fractions like $\frac{5}{x-1}+\frac{2}{x+3}$. You find an LCD and multiply each term by the missing factor. $\frac{5}{x-1}\left(\frac{x+3}{x+3}\right)+\frac{2}{x+3}\left(\frac{x-1}{x-1}\right)=\frac{5x+15+2x-2}{(x-1)(x+3)}=\frac{7x+13}{x^2+2x-3}$.

However, at times, there is a necessity to reverse the process: to change $\frac{7x+13}{x^2+2x-3}$ to $\frac{5}{x-1}+\frac{2}{x+3}$. This is called partial fraction decomposition. The method is called the "Heaviside Method." (Oliver Heaviside -1850).

The way to begin is to factor the denominator: $\frac{7x+13}{x^2+2x-3}=\frac{7x+13}{(x-1)(x+3)}$.

Realize that if this were to be written as the sum of two fractions, there must be two numbers A and B such that $\frac{7x+13}{(x-1)(x+3)}=\frac{A}{x-1}+\frac{B}{x+3}$.

To find A

Cover up the $(x-1)$ in $\frac{7x+13}{(x-1)(x+3)}$

giving $\frac{7x+13}{(x+3)}$

Plug $x=1$ in: $A=\frac{7(1)+13}{1+3}=\frac{20}{4}=5$

To find B

Cover up the $(x+3)$ in $\frac{7x+13}{(x-1)(x+3)}$

giving $\frac{7x+13}{(x-1)}$

Plug $x=-3$ in: $B=\frac{7(-3)+13}{-3-1}=\frac{-8}{-4}=2$

$$\boxed{\frac{7x+13}{(x-1)(x+3)}=\frac{5}{x-1}+\frac{2}{x+3}}$$

2) Write the partial fraction decomposition for:

a) $\frac{x-19}{x^2+4x-5}$

b) $\frac{4x-9}{x^2-3x}$

c) $\frac{4x+10}{x^2-14x+48}$

d) $\frac{-5x^2-4x-4}{x^3-4x}$

Note that this technique will not work if the denominator has repeated factors. The technique to decompose expressions with denominators like $x^2-4x+4=(x-2)^2$ is slightly different.

Although it is not part of the AP precalculus curriculum, let show an example. We will decompose $\dfrac{2x}{x^2-4x+4}$.

$\dfrac{2x}{x^2-4x+4}=\dfrac{2x}{(x-2)^2}=\dfrac{A}{x-2}+\dfrac{B}{(x-2)^2}$. One of the denominators gets an $x-2$ and the other $(x-2)^2$.

Multiplying out by the LCD $(x-2)^2$, we get $2x=A(x-2)+B$
As before, let $x = 2$ and we get $B = 4$.
To find A, let x equal any number other than 2. Let's say $x = 3$. So $6 = A + 4$ and $A = 2$.

So $\dfrac{2x}{x^2-4x+4}=\dfrac{2}{x-2}+\dfrac{4}{(x-2)^2}$.

e) Perform partial fraction decomposition on $\dfrac{12x+43}{x^2-8x+16}$

The Binomial Theorem

Many times we have to expand expressions in the form $(x+y)^n$. This is called binomial expansion. When doing so, we get a sum involving terms in the form of ax^by^c where the exponents b and c are non-negative integers and $b + c = n$. For example $(x+y)^3=x^3+3x^2y+3xy^2+y^3$. The coefficients for each term can be found directly using combination theory, but in a typical precalculus course, we just look at Pascal's Triangle. This was named for the 17th-century French mathematician Blaise Pascal, but it goes back to China in the 11th century.

Any row is found by adding the two numbers just above to the left and right of each position in the triangle. The first row gives the coefficients for the expansion $(x+y)^0$ which is 1. The second row gives the coefficients for $(x+y)^1=1x+1y$. The 3rd row gives the coefficients for $(x+y)^2=1x^2+2xy+1y^2$. And so on.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1

If we are expanding $(x-y)^n$, then starting with the 2nd term, every other term is negative.

3) Expand the following:

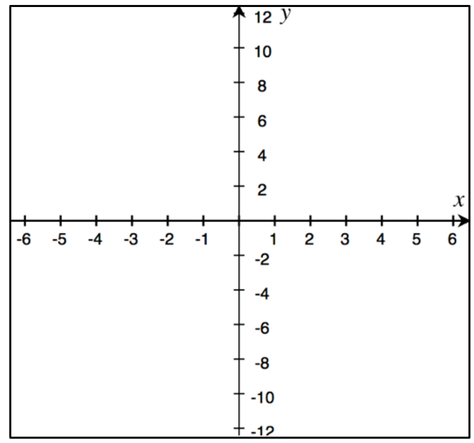
a) $(x+y)^4$ b) $(x-y)^4$ c) $(x+2)^5$

d) $(2x-3)^4$ e) the middle term of $(3x^2-2)^6$

Topic 1.9 – Rewriting Expressions – Homework

1. For each of the following rational functions, find the requested information and sketch. Calculator verify.

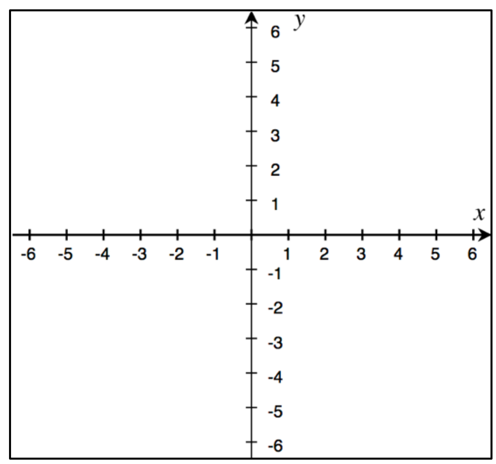
a. $y = \dfrac{x^2 - 5x - 6}{x - 3}$



a. domain:
b. VA:
c. OA:
d. Zeros
e. Interval Work: ____________________

f. $y -$ intercept:

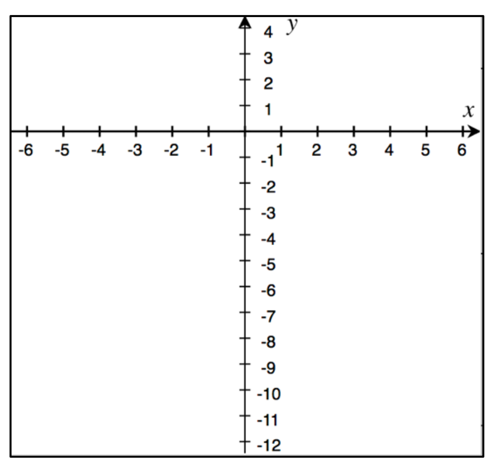
b. $y = \dfrac{x^2 + 3x}{x + 2}$



a. domain:
b. VA:
c. OA:
d. Zeros
e. Interval Work: ____________________

f. $y -$ intercept:

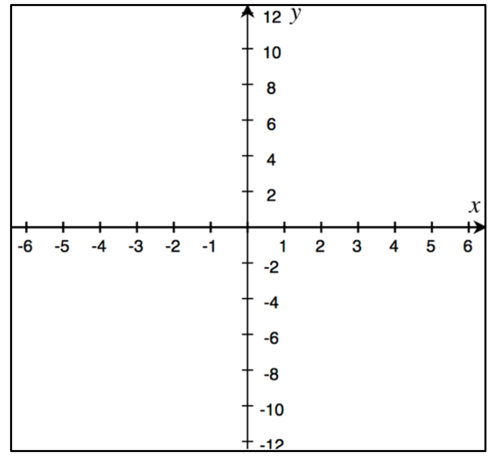
c. $y = \dfrac{2x^3}{x^2 + x}$



a. domain:
b. VA:
c. OA:
d. Zeros
e. Interval Work: ____________________

f. $y -$ intercept:

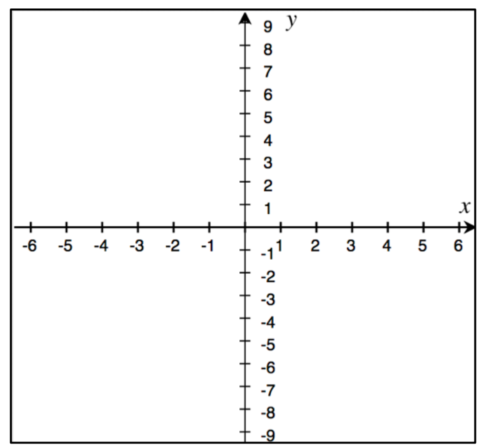
d. $y = \dfrac{x^3 - 3x + 2}{x^2 - 2x}$



a. domain:
b. VA:
c. OA:
d. Zeros
e. Interval Work: ____________________

f. $y -$ intercept:

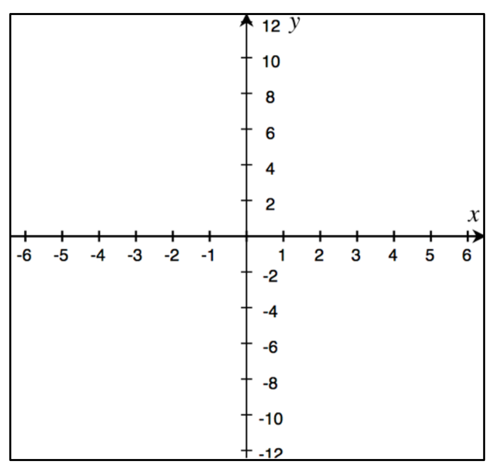
e. $y = \dfrac{x^3 + x^2 - 2}{x + 1}$



a. domain:
b. VA:
c. OA:
d. Zeros
e. Interval Work: ____________________

f. $y -$ intercept:

f. $y = \dfrac{x^3 - 2x^2 - 4x + 8}{x - 1}$



a. domain:
b. VA:
c. OA:
d. Zeros
e. Interval Work: ____________________

f. $y -$ intercept:

2. Perform partial fraction decomposition on the following.

a. $\dfrac{5x+55}{x^2+5x}$

b. $\dfrac{3x-57}{x^2-8x+7}$

c. $\dfrac{17x-53}{x^2-2x-15}$

d. $\dfrac{9x-96}{x^2-18x+80}$

e. $\dfrac{-2x^2+31x-12}{x^3+x^2-12x}$

f. $\dfrac{3x+16}{x^2+12x+36}$

3. Expand the following:

a. $(x+5)^3$

b. $(x-5)^3$

c. $\left(2x-\dfrac{1}{2}\right)^4$

d. $\left(\pi-x^2\right)^5$

e. the two middle terms of $(2x-3)^7$